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Singularity confinement analysis of integrodifferential equations of Benjamin–Ono type

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Abstract. We analyse the integrability of various types of intregrodifferential equations belonging to the Benjamin–Ono family. The method used is a combination of two well known integrability detectors: the Painlevé method (for continuous systems) and the singularity confinement (for discrete systems). We confirm the results of Hietarinta based on the study of multisoliton solutions. In particular, we show that the third-order extension to Benjamin–Ono that he proposed does pass the test and is thus an excellent candidate for integrability.

Integrable equations involving the Hilbert transform have been known since the 1970s, the Benjamin–Ono (BO) equation [1] being the prototype,

$$u_t + 2uu_x + Hu_{xx} = 0 \tag{1}$$

where the Hilbert transform is given by

$$Hf(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(z)}{z-x}\,\mathrm{d}z. \tag{2}$$

Related to BO is the intermediate-long wave equation (ILW) [2] involving also an integral transform

$$u_t + h^{-1}u_x + 2uu_x + Tu_{xx} = 0 \tag{3}$$

where the 'coth' transform is given by

$$Tf(x) = \frac{1}{2h} P \int_{-\infty}^{\infty} \coth\frac{\pi(z-x)}{2h} f(z)\,\mathrm{d}z. \tag{4}$$

The integrability of these equations is already established in the sense that their Lax pair is known [3, 4] and the inverse scattering transform (IST) has been performed [5].

However, these equations present a difficulty as far as the Painlevé integrability criterion is concerned. The latter requires that for all integrable equations the solutions be free of movable critical singularities. 'Movable' in this last sentence means a manifold that depends on the initial data (and the further requirement is that it be non-characteristic), while 'critical' is a singularity that induces multivaluedness. The Painlevé test is a local one, it examines locally the behaviour at each singularity taken in isolation. Since BO-type equations are non-local by construction one can wonder how one can apply this local integrability criterion to them.

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A first answer to this puzzle was presented in [6]. It was shown that the BO equations can be written as a system of differential equations in $(2+1)$ dimensions complemented by a boundary condition. We have, in fact, a first equation that is simply linear (Laplace's equation),

$$u_{xx} + u_{yy} = 0 \tag{5a}$$

with boundary condition $u = f(x, t)$ at $y = 0$ and $\partial u/\partial y = 0$ at $y = -h$ for ILW or $u = 0$ at $y = -\infty$ for BO. Written in $(2+1)$ dimensions the BO/ILW equations assume the form $(5a, b)$ with

$$u_t + 2uu_x + u_{xy} = 0 \qquad \text{at } y = 0. \tag{5b}$$

Since in this formulation the BO equations are purely local, the Painlevé criterion can be (and has been) applied in a straightforward way.

However, this trick was not the final solution to the problem. The essential difficulty can be understood if one looks at the bilinear formulation of the BO equation [7]. It is well known that BO can be written as

$$(\mathrm{i}D_t - D_x^2)F_+ \cdot F_- = 0 \tag{6}$$

where D is the Hirota operator defined through its action on the dot product $Df \cdot g = (\partial_x - \partial_{x'}) f(x) g(x')|_{x=x'} = f'(x)g(x) - f(x)g'(x)$. The τ-functions $F_\pm$ are defined in the upper(lower) complex x-plane. Thus BO relates the function at a point in the upper half-plane to a point in the lower half-plane. From the point of view of singularity analysis these two functions are completely independent: the locations of the singularities in the upper and lower half-planes are not *a priori* related. Thus we have, in some sense, only one equation with *two* unknown functions and the singularity analysis, cannot be carried out. Similarly, in the case of ILW we have

$$(\mathrm{i}D_t + \mathrm{i}h^{-1}D_x - D_x^2)F_+ \cdot F_- = 0 \tag{7}$$

where $F_\pm = F(x \pm \mathrm{i}h, t)$ with h real. The non-locality is due to the fact that (7) relates the function at a point $(x + \mathrm{i}h)$ to the function at the point $(x - \mathrm{i}h)$. Again, for the traditional singularity analysis F_+ and F_- are considered as different objects and, again, we have only one equation for *two* unknowns. The way out of this deadend is based on the observation that F is defined (in the complex x-plane) in strips parallel to the real x-axis of width $2h$. Thus (7) can be viewed as a mapping relating the Fs in two adjacent strips. We have proposed this interpretation in [8] and it made possible the treatment of the equations of the BO family as differential-difference systems.

Let us illustrate this approach though a simple examples based on the ILW. Before proceeding further we remark that for the implementation of singularity analysis we can introduce a new t-variable in order to absorb the $\mathrm{i}h^{-1}D_x$ term and bring the equation to the form

$$(\mathrm{i}D_t - D_x^2)F_+ \cdot F_- = 0 \tag{8}$$

thus formally equivalent to BO (although the two remain different as far as IST is concerned). Instead of working out this (after all) uninteresting example let us investigate what are the source terms compatible with integrability:

$$(D_t + D_x^2 + \mu(x, t))F_+ \cdot F_- = 0 \tag{9}$$

where the t variable has again been transformed so as to absorb the i factor. We recall that the τ-functions F can only have zeros: critical singularities may appear only as a consequence of a vanishing τ-function. Let us, for the sake of simplicity in notation, use the symbols F, G and H for the τ-functions at the points $(x - 3ih)$, $(x - ih)$ and $(x + ih)$, respectively. The idea is that F is regular around some singularity manifold $\phi = 0$ on which G has a simple zero. The Painlevé property requirement [9] is that this vanishing G do not induce a critical singularity on H (or subsequent τ-functions). Moreover, since the system is a differential-*difference* one, the singularity confinement requirement [10] is that the singularity, i.e. the vanishing of G, does not propagate indefinitely with the mapping (9) from one strip to the other but disappears after some iteration. Here we will ask that H not only have no logarithmic singularities but also be finite on the manifold $\phi = 0$. In order to implement the singularity analysis algorithm, we start with a regular F which can be expanded in powers of ϕ as $F = \sum_{k=0} F_k \phi^k$ while G has the expansion $G = \sum_{k=1} G_k \phi^k$. From (9), rewritten in explicit form as

$$F_{xx}G - 2F_xG_x + FG_{xx} + F_tG - FG_t + \mu FG = 0 \tag{10}$$

we can compute the G_k in terms of the F_k. The 'resonances' [11] of this equation being -1 and 0, we have ϕ and G_1 as free functions. (The calculation is greatly simplified if one uses the Kruskal [12] ansatz: $\phi = x + f(t)$.) Next we write the equivalent to (10) relating G and H. The resonances are 0 and 3 and thus, if we expand $H = \sum_{k=0} H_k \phi^k$, H_0 and H_3 must be free. This is not automatically true for H_3 and a condition must be satisfied. Computing H_1 and H_2 we find that the resonance condition is indeed satisfied for any choice of μ, and therefore the ILW/BO equation with source term $\mu(x, t)$ is integrable for any source μ. Thus, the combination of Painlevé analysis with singularity confinement makes possible the treatment of integrodifferential equations of the BO family.

To be fair, we must point out here that the analysis we presented was limited to the examination of confinement in one step. In particular, we have assumed that H was finite. However, given the form of equation (10), H can have also a triple zero, i.e. $H \propto \phi^3$. If we assume this leading behaviour for H we find out that the confinement of this singularity requires (at minimum two) additional steps. An exhaustive analysis should consider the case of confinement after an arbitrary number of steps. If, for instance, at some stage we have a singular G with leading behaviour $G \propto \phi^n$ with $n = m(m+1)/2$ with integer m then the leading behaviour of H is $H \propto \phi^k$ with either $k = m(m-1)/2$ or $k = (m+1)(m+2)/2$. The first corresponds to a less singular behaviour while the second is a more singular one. Following the first, we move towards confinement and it is important to note that for equation (10) this possibility always exists. Although the pattern of confined singularities may be fairly complicated, it is our experience that the simplest singularity pattern contains already the essential integrability constraints.

Another, well known, integrability criterion for partial differential equations (PDEs) is the existence of multisoliton solutions. In a series of papers [13], Hietarinta has identified bilinear PDEs belonging to various families (KdV, mKdV, SG, NLS, BO) that possess multisoliton solutions. The analysis was performed up to the level of the first non-trivial, usually three-, but sometimes four-, soliton solution. These results confirmed the integrability of known cases and suggested the integrability of some new ones. The results of the multisoliton study were strengthened in [14] where we have shown that the equations that possess multisoliton solutions also satisfy the Painlevé integrability criterion. In that study the analysis of the BO-type equation was missing since, at that time, their singularity confinement approach had not yet been developed. In this paper we shall complete the comparison of the two methods by presenting the confined singularity analysis of the BO equations of Hietarinta. We shall not go into all the technical details but limit ourselves to the results.

The first equation identified is

$$(D_x D_t + aD_x + bD_t)F_+ \cdot F_- = 0 \tag{11}$$

which is a well known integrable equation due to Matsuno [15]. The appearance of the $aD_x + bD_t$ term suggests that a D_y term might be compatible with integrability. However, it turns out that this equation does not have multisoliton solutions. The confined singularity results are *exactly* the same: Matsuno's equation does pass the test, while the $(2 + 1)$-dimensional extension fails.

The second equation is

$$(D_x^3 + D_x^2 + D_t)F_+ \cdot F_- = 0 \tag{12}$$

which was obtained for the first time by Hietarinta [13]. The confined singularity analysis can be applied along the lines similar to that of (9). (However, here we have two simple behaviours: G vanishing like ϕ or ϕ^2. The former has resonances 0,1 and 5 while the latter has only 0 and 2 as positive resonances.) The result is that (12) does pass the test and thus we expect the Hietarinta-BO equation to be integrable.

Another interesting equation that we will analyse here is the intermediate nonlinear Schrödinger equation (INLS) proposed recently by Pelinovsky [16] in order to describe internal waves in a two-layer stratified fluid. Written schematically, this equation assumes the form

$$\mathrm{i}u_t = u_x x + u(\mathrm{i} + T)|u|_x^2 + \mu u \tag{13}$$

where T is the coth-integral operator (2) appearing in the ILW equation and $\mu(x, t)$ is a source term. The bilinearization of the INLS equation was performed by Pelinovsky himself who rewrote (13) (in a frame propagating with velocity c) as

$$(\mathrm{i}D_t + \mathrm{i}cD_x + D_x^2 + \mu)F \cdot G = 0 \tag{14a}$$

$$(\mathrm{i}D_t + \mathrm{i}cD_x + D_x^2 + \mu)\overline{G} \cdot \overline{F} = 0 \tag{14b}$$

$$\mathrm{i}D_x F \cdot \overline{F} + c(G\overline{G} - F\overline{F}) = 0. \tag{14c}$$

Furthermore, he remarked that these equations were closely related to the bilinear Bäcklund transforms of BO proposed by Nakamura [17]:

$$(\mathrm{i}D_t - 2\mathrm{i}\lambda D_x - D_x^2 + \mu)F \cdot G = 0 \tag{15a}$$

$$(\mathrm{i}D_t - 2\mathrm{i}\lambda D_x - D_x^2 + \mu)\overline{F} \cdot \overline{G} = 0 \tag{15b}$$

$$(D_x + \mathrm{i}\lambda)F \cdot \overline{G} = \mathrm{i}\nu \overline{F}G \tag{15c}$$

where λ, μ and ν are constants. Here is their logical link that will also lead to a straightforward implementation of the singularity confinement test: we start from (14a) for F, G in a given strip and look for $\overline{F}, \overline{G}$ in the next strip satisfying (14b) and (14c); but (14b) is just the upshift of (14a) provided we permute $\overline{F}$ and $\overline{G}$. In this case (14c) becomes

$$\mathrm{i}D_x F \cdot \overline{G} + c(G\overline{F} - F\overline{G}) = 0 \tag{16}$$

which is contained in Nakamura's (15c). Thus, it suffices to analyse Nakamura's equation which we rewrite as

$$(D_t + D_x^2 + \mu)F \cdot G = 0 \tag{17a}$$

$$(D_t + D_x^2 + \overline{\mu})\overline{F} \cdot \overline{G} = 0 \tag{17b}$$

$$D_x F \cdot \overline{G} + \lambda(G\overline{F} - F\overline{G}) = 0 \tag{17c}$$

where we have redefined x and t so as to absorb the i's and the λD_x term in equations (17a, b). Moreover, we have limited ourselves to $\nu = \lambda$ and introduced a source term

μ which we allow to depend on the number of the strip we are in (clearly an unphysical assumption but one which makes sense in as far as (17) is viewed just as a differential-difference system). We remark readily that (17c) allows one to compute explicitly $\overline{F}$ from the knowledge of F, G and $\overline{G}$.

The implementation of singularity analysis to the system (17) is straightforward. We consider that the τ-functions $\underline{F}$ and $\underline{G}$ (obeying the down-shift of equation (17a)) are regular. The singularity corresponds to the vanishing of either F or G on some singularity manifold $\phi = 0$. Given the form of equations (17a) and the down-shift of (17c), linking F and G to the regular $\underline{F}$ and $\underline{G}$, we find that F and G can only vanish linearly with ϕ. The singularity confinement requirement is that both $\overline{F}$ and $\overline{G}$ be regular. Using (17c) we can eliminate $\overline{F}$ and obtain a third-order equation for $\overline{G}$. Using this equation we find that confinement in one step is always possible. Indeed, when $G \propto \phi$ we find that $\overline{G}$ behaves like ϕ^0 or ϕ^3. The first case corresponds to a regular $\overline{G}$, provided no logarithmic singularities appear. In order to check this we compute the resonances associated with this behaviour. We find $r = 0, 2, 3$ and, moreover, the resonance conditions are satisfied. Similarly, when $F \propto \phi$ we find that G behaves like ϕ^0 or ϕ^1. The first case has resonances $r = 0, 1, 2$ and the resonance condition at $r = 1$ is $\overline{\mu} = \mu$. Thus, for integrability, the source term cannot depend on the strip we are in. We must point out here that the behaviour $\overline{G} \propto \phi^3$ or $\overline{G} \propto \phi$ corresponds to a singularity that does not confine in one step. The condition for the absence of logarithmic singularities for these behaviours is, again, $\overline{\mu} = \mu$. The study of the confinement of these singularities should, in principle, be pursued further. In fact, as we have explained earlier we must, for completeness, consider all the patterns of confinement in any number of steps. However, as expected, the confinement in one step furnishes the essential integrability constraint $\overline{\mu} = \mu$. Thus with this condition the Nakamura–Pelinovsky equations are integrable.

To summarize, we remark that the combination of Painlevé analysis and the singularity confinement approach is a most adequate tool for the investigation of the BO-type integrodifferential equations. The key to this application is the interpretation of the ILW equation as a differential-difference equation. In this paper we have shown that the ILW equation can be integrably extended so as to include a source term. The results of Hietarinta (based on the study of multisoliton solutions) on the Matsuno equation and on the extension of BO through higher-order terms were confirmed by our analysis. In particular, the Hietarinta-BO equation must be a new integrable equation. One important remark is that the use of a 'discrete' method (the singularity confinement) is essential for the investigation of the equations of the BO family. These equations are non-local and a continuous approach is just inadequate for their treatment. Finally, we should point out that the use of the bilinear formalism was of capital importance in our approach: it makes the application of the singularity confinement algorithm really straightforward.

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